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Packet Radio Temporary Note #135
March 18, 1975

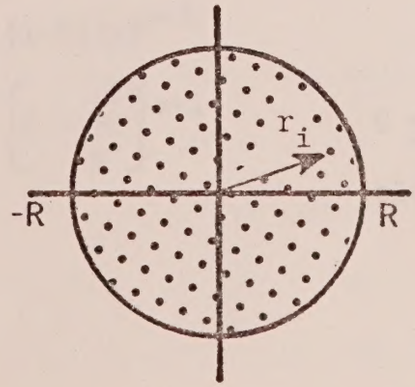
ON A PROPERTY OF RANDOM NETS*

I. INTRODUCTION

The purpose of this note is to derive an interesting result dealing with the distance to the nearest point in a field of random points. We show that the probability density of the distance to the nearest point in such a field is Rayleigh, and that the parameter of the Rayleigh density is proportional to the average number of random points per unit area. This result seems to have applications in the study of packet radio nets, networks of small earth stations for satellite systems and in the placement of exchanges in a telephone network. One simple example of such an application is provided.

II. DERIVATION

Consider a two dimensional plane containing random points sprinkled throughout the plane with a uniform density of ρ points per unit area. Without loss of generality we select a large circle of radius R about the origin, and let n be the number of points inside the circle.



*The work reported here was performed at the DCA, Defense Communications Engineering Center.

Let the radii of the n points be $r_1, r_2, \ldots, r_n$. We are interested in deriving the density of

$$r = \min \{r_1, r_2, \ldots, r_n\} \tag{1}$$

It is clear that if we select the circle so that n is large enough, the density of r will not depend upon R or n. For the case where the points are distributed uniformly over the plane the density of r_i is

$$p(r_i) = \begin{cases} \dfrac{2r_i}{R^2} & 0 \leq r_i \leq R \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

Define

$$P(r_i) = \int_0^{r_i} p(x)\, dx = \frac{r_i^2}{R^2} \tag{3}$$

Then the density of r is

$$\begin{aligned} q(r) &= np(r)\,[1-P(r)]^{n-1} \\ &= \begin{cases} \dfrac{2nr}{R^2}\left[1 - \dfrac{r^2}{R^2}\right]^{n-1} & 0 \leq r \leq R \\ 0 & \text{elsewhere} \end{cases} \end{aligned} \tag{4}$$

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But for n large, we have

$$n = \pi R^2 \rho \tag{5}$$

and using (5) to substitute for R^2 in (4), we have

$$q(r) = \begin{cases} 2\pi\rho r\left[1 - \frac{\pi\rho r^2}{n}\right]^{n-1} & 0 \leq r \leq R \\ 0 & \text{elsewhere} \end{cases} \tag{6}$$

and in the limit as $n \to \infty$

$$q(r) = 2\pi\rho\, r\, e^{-\pi\rho r^2} \tag{7}$$

a Rayleigh density with parameter $[2\pi\rho]^{-1}$. Therefore

$$E[r] = \frac{1}{2\sqrt{\rho}} \tag{8}$$

III. EXAMPLE

To show how we might apply this result consider the question of how many earth stations to use in a satellite communication network where we assume both the network users and the earth stations are randomly distributed on a plane according to a uniform distribution. We can think of two extremes in the design of such a network - a low density of earth stations and a high density of earth

stations. In the former case we may find that the cost of the land lines needed to connect the users to the few earth stations is too large, while in the latter case we may find that the cost of providing an excessive number of earth stations predominates.

There are two reasons to analyze this artificial problem. First, the uniform distribution of users may not be too unrealistic a model for some large networks. Second, the uniform assumption provides an upper bound to cost estimates since the bunching of users and the placement of earth stations to take advantage of such bunching can only serve to decrease the total network cost.

We define:

C = total cost per unit area of the earth stations plus the lines from the users to the earth stations ($/sq. mile)

ρ_u = total density of users (users/sq. mile)

ρ_s = total density of earth stations (stations/sq. mile)

L = cost per unit distance of the lines ($/mile)

s = earth station cost ($)

Then we have, using (8)

$$C = \rho_s s + \frac{1}{2\sqrt{\rho_s}} \rho_u \tag{9}$$

we may take a derivative of (9) with respect to ρ_s to find the optimum density of earth stations.

$$(\rho_s)_{opt.} = \left[\frac{\rho_u L}{4s}\right]^{2/3} \tag{10}$$

and if we assume an average user density of 10 users/sq. mile, a line cost of $100/mile and an earth station cost of $100,000 per station we obtain

$$(\rho_s)_{opt.} \approx \frac{1}{50}$$

or about one earth station every 50 square miles.

For the case of a system the size of DCS, with about 30,000 user terminals over an area the size of CONUS (4.5×10^6 square miles), we have

$$\rho_1 = 7 \times 10^{-3} \frac{\text{users}}{\text{sq. mile}}$$

Taking 1 equal to $240 per mile and s equal to 10^6 per earth station we have

$$(\rho_s)_{opt.} = 56 \times 10^{-6}$$

or about 250 earth stations distributed within CONUS.